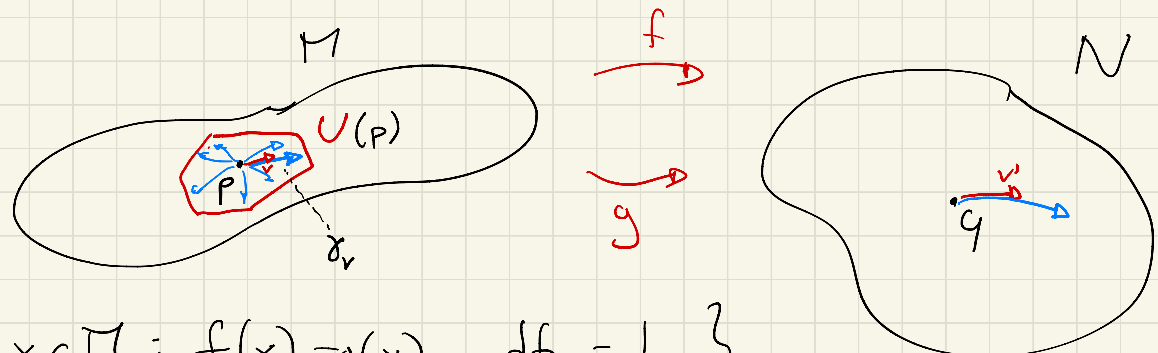
Lezione 28

Prop: M, N pR $\quad f,g : M^n \to N^n$ isometrie $\quad M$ connessa.

Se $\exists\, p \in M$ t.c. $f(p) = g(p)$ & $df_p = dg_p : T_pM \to T_{f(p)}N$

Allora $f \equiv g$.

dim:



$U \subseteq M \qquad U = \{x \in M : f(x) = g(x),\ df_x = dg_x\}$

$p \in U \Rightarrow U \neq \emptyset \qquad$ U chiuso (facile: condizioni chiuse lette in carte)

U aperto (questo basta $\Rightarrow U = M$)

$p \in U \Rightarrow \exists U(p) \subseteq U$

Prendo $U(p)$ intorno normale di p

Unicità delle geodetiche $\Rightarrow$ $f|_{U(p)} \equiv g|_{U(p)} \Rightarrow df_x = dg_x$
$\forall x \in U(p)$

$\Rightarrow U(p) \subseteq U$

$\square$

Cor: $\mathrm{Isom}(\mathbb{R}^{p,q}) = \{ x \mapsto Ax + b \mid A \in O(p,q),\ b \in \mathbb{R}^{p,q} \}$

$\mathrm{Isom}(S^{p,q}) = O(p+1,q)$ $S^{p,q} \subseteq \mathbb{R}^{p+1,q}$

$\mathrm{Isom}(H^{p,q}) = O(p,q+1)$ $H^{p,q} \subseteq \mathbb{R}^{p,q+1}$

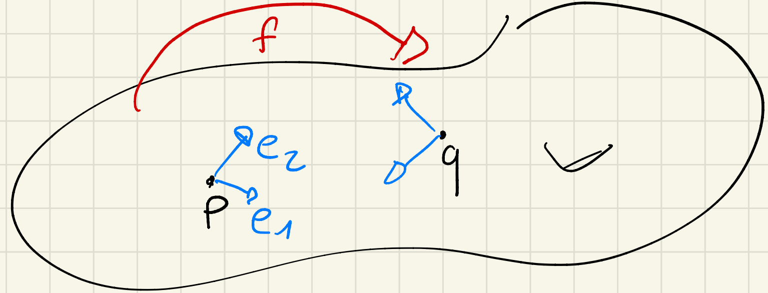
dim: Già visto $\boxed{\supseteq}$

Es: I gruppi di destra agiscono transitivamente sui frame. $:= \{ p, e_1, \dots, e_n$ base ort. in $T_pM \}$

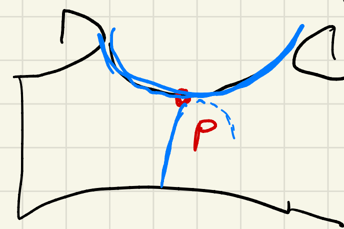
Rigidità $\Longrightarrow$ $\boxed{=}$

$\Downarrow$ $\quad$ $\Uparrow$

Le isometrie agiscono in modo libero sui frame



CURVATURA



$S \subseteq \mathbb{R}^3$

Def: (M, ∇) Il TENSORE DI RIEMANN è un campo tensoriale R di tipo $(1,3)$

cioè $\quad R(p) : T_pM \times T_pM \times T_pM \to T_pM$

$$u, v, w \in T_pM$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$X, Y, Z \in \mathfrak{X}(U(p))$$

$$\boxed{R(p)(u, v, w) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$T_{ij}^{k} = \Gamma_{ij}^{k} - \Gamma_{ji}^{k}$

Prop: È ben definito.

dim: In coordinate

$$\nabla_X \nabla_Y Z = \nabla_X \left(Y^i \frac{\partial Z^k}{\partial x^i} e_k + Y^i Z^j \Gamma_{ij}^{k} e_k \right)$$

$$= \underset{(1)}{X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial Z^k}{\partial x^i} e_k} + \underset{(2)}{Y^i X^j \frac{\partial^2 Z^k}{\partial x^j \partial x^i} e_k} + \underset{(3)}{Y^i \frac{\partial Z^k}{\partial x^i} X^l \Gamma_{lk}^{m} e_m}$$

$$\underset{(4)}{X^l \frac{\partial Y^i}{\partial x^l} Z^j \Gamma_{ij}^{k} e_k} + \underset{(5)}{Y^i X^l \frac{\partial Z^j}{\partial x^l} \Gamma_{ij}^{k} e_k} + \underset{(6)}{Y^i Z^j X^l \frac{\partial \Gamma_{ij}^{k}}{\partial x^l} e_k}$$

$$+ \underset{(7)}{Y^i Z^j \Gamma_{ij}^{k} X^l \Gamma_{lk}^{m} e_m}$$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \underbrace{(1) + (4)} + (6) + (7)$$

$$(1)+(4) = \nabla_{[X,Y]} Z$$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = (6)+(7)$$ non dipende da estensione $\square$

Durante la dim abbiamo visto che

$$R_{ijk}{}^{e} = \text{coordinate di } R = R(e_i, e_j, e_k)^{e}$$

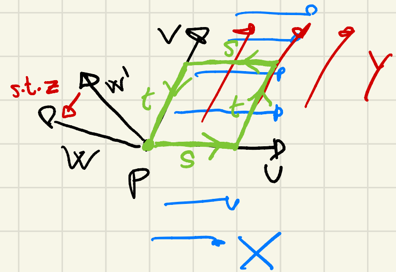
$$R(u,v,w)^{e} = R_{ijk}{}^{e} u^i v^j w^k$$

$$R(u,v,w) = R_{ijk}{}^{e} u^i v^j w^k \mathbf{e}_e$$

$$\boxed{R_{ijk}{}^{e} = \frac{\partial \Gamma_{jk}^{l}}{\partial x_i} - \frac{\partial \Gamma_{ik}^{l}}{\partial x_j} + \Gamma_{im}^{l} \Gamma_{jk}^{m} - \Gamma_{jm}^{l} \Gamma_{ik}^{m}}$$

Esempio: $\mathbb{R}^{p,q}$ $\quad \Gamma_{ij}^{k} \equiv 0 \implies R \equiv 0$

Teo:



$\gamma_{s,t} \quad \forall s,t>0$

$z = R(u,v,w)$

In carte $\quad w' = w - R(p)(u,v,w)\,st + o(s^2+t^2)$

dove $w \in T_pM$ e $w' = h_{s,t}(w)$ $\quad h_{s,t}: T_pM \to T_pM$ isometria

$h_{s,t} = \Gamma(\gamma_{s,t})$

$$\boxed{R_{ijk}^{\ell} = \frac{\partial \Gamma_{jk}^{\ell}}{\partial x_i} - \frac{\partial \Gamma_{ik}^{\ell}}{\partial x_j} + \Gamma_{im}^{\ell}\Gamma_{jk}^{m} - \Gamma_{jm}^{\ell}\Gamma_{ik}^{m}}$$

In coordinate normali: (M, g)

$\odot\ \Gamma_{ij}^{k}(0) = 0 \qquad \odot\ g_{ij}(0) = \eta_{ij} = \begin{pmatrix} -I_q & \\ & I_p \end{pmatrix} \qquad \odot\ \dfrac{\partial g_{ij}}{\partial x_k} = 0$

$$\odot\ \boxed{R_{ijk}{}^{l}(0) = \frac{\partial \Gamma_{jk}^{l}}{\partial x_i} - \frac{\partial \Gamma_{ik}^{l}}{\partial x_j}}$$

Prop: $g_{ij}(x) = \eta_{ij} + \frac{1}{3} R_{ijkl}(0)\, x^k x^l + o(\|x\|^2)$

$R_{ijkl} := R_{ijk}{}^{m} g_{ml}$ versione $(0,4)$

$$\odot\ R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 g_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l} \right)$$

Prop: R ha queste simmetrie:

1) $R_{ijkl} = -R_{jikl} = -R_{ijlk}$

2) $R_{ijkl} = R_{klij}$ $\qquad R(u,v,w,z) = R(w,z,u,v)$

$\forall u,v,w,z \in T_pM$

3) $R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} = 0$

Prop: I tensori $(0,4)$ in $\mathbb{R}^n$ che soddisfano (1), (2), (3) formano un sottospazio di $T_0^4(\mathbb{R}^n)$ di dim

n^4

$$\frac{1}{12} n^2 (n^2 - 1)$$

$\boxed{R_{1212}}$

$n=2$: 1

$n=3$: 6

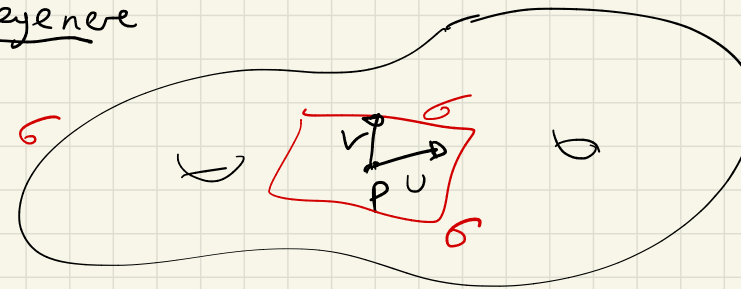
$n=4$: $\boxed{20}$

Prop: $\nabla_a R_{ijk}{}^{l}$ è $(1,4)$

IDENTITÀ DI BIANCHI: $\nabla_a R_{ijk}{}^{l} + \nabla_i R_{jak}{}^{l} + \nabla_j R_{aik}{}^{l} = 0$

CURVATURA SEZIONALE

(M,g) pR $\qquad p \in M \qquad \sigma \subseteq T_pM$ piano vettoriale non degenere



$K(\sigma)$ CURVATURA SEZIONALE LUNGO σ

$$K(\sigma) := \frac{R(p)(u,v,u,v)}{Q(u,v)}$$

$$Q(u,v) = \langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2 \neq 0$$

Se g def+ $\qquad Q = \left(\text{Area } \right.$  $\left.\right)^2$

Prop: $K(\sigma)$ non dipende dalla base u, v per σ

dim:

$$(u,v) \to \begin{cases} (v,u) \\ (u, v+\lambda u) \\ (\lambda u, v) \end{cases}$$

esercizio

Prop: R determina K e K determina R

TENSORE DI RICCI

$$(M, \nabla) \dashrightarrow R_{ijk}{}^{l} \to \begin{cases} R_{ijk}{}^{i} \\ R_{ijk}{}^{j} \\ R_{ijk}{}^{k} \end{cases}$$

Def: Il TENSORE DI RICCI è $R_{ij} = R_{kij}{}^{k}$

campo tensoriale $(0,2)$

Ric Riem

Prop: R_{ij} è simmetrico

dim: $R_{ij} = R_{kij}{}^{k} = R_{kijl}\, g^{kl} =$

$$R_{ji} = R_{kji}{}^{k} = R_{kjil}\, g^{lk}$$

Prop: In coordinate normali

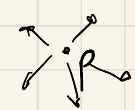
$$\det g_{ij} = \det \eta \left(1 - \frac{1}{3} R_{ij}(0)\, x^i x^j \right) + o(\|x\|^2)$$

CURVATURA SCALARE

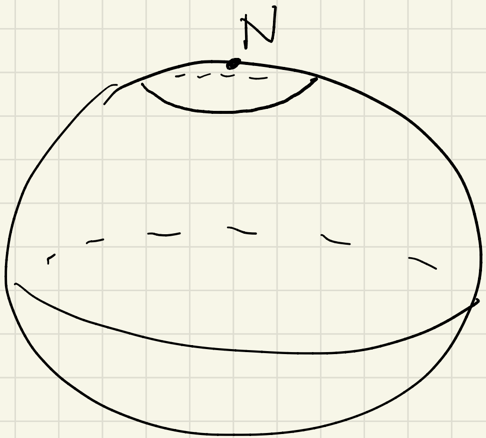
$R = R_{ij}\, g^{ij}$

CURVATURA SCALARE

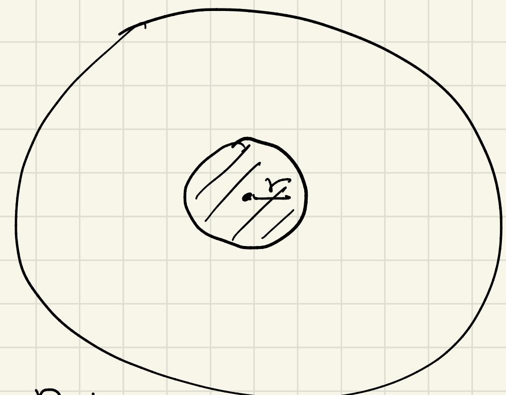
$R \in \mathcal{C}^\infty(M)$



Prop: $$\mathrm{Vol}\,(B(p,r)) = \underbrace{V_n(r)}_{\text{volume euclideo di } B(0,r)\,\subseteq\,\mathbb{R}^n} \left(1 - \frac{1}{6(n+2)} R(p)\, r^2 + o(r^3)\right)$$



R>0



R<0